Linear determining equations, differential constraints and invariant solutions.

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Abstract

A construction of differential constraints compatible with partial differential equations is considered. Certain linear determining equations with parameters are used to find such differential constraints. They generalize the classical determining equations used in the search for admissible Lie operators. As applications of this approach non-linear heat equations and Gibbons-Tsarev's equation are discussed. We introduce the notion of an invariant solution under an involutive distribution and give sufficient conditions for existence of such a solution.

PACS numbers: 02.30.Jr, 02.30.Ik, 44.05.+e
Mathematics Subject Classification: 34G20, 35K57

1. Introduction.

As is well known, one can produce many exact solutions of partial differential equations by means of additional constraints [1], [2]. Differential constraints arisen originally in the theory of partial differential equations of the first order. Lagrange in particular used differential constraints to find the total integral of nonlinear equation

$$F(x, y, u, u_x, u_y) = 0.$$

Darboux [3] applied differential constraints to integrate the partial differential equations of the second order. The detailed description of the Darboux method can be found in [1], [4].

The general formulation of the method of differential constraints requires that the original system of partial differential equations

$$F^1 = 0, ..., \quad F^m = 0 \tag{1.1}$$

be enlarged by appending additional differential equations (differential constraints)

$$h_1 = 0, ..., \quad h_p = 0, \tag{1.2}$$

such that the over-determined system (1.1), (1.2) satisfies some conditions of compatibility.

The theory of over-determined systems was developed by Delassus, Riquier, Cartan, Ritt, Kuranishi, Spencer and others. One can find references in the book of Pommaret [5]. Now the applications of over-determined systems include such diverse fields as differential geometry, continuum mechanics and nonlinear optics. Unfortunately the problem of finding all differential constraints compatible with certain equations can be more complicated than the investigation of the original equations. Therefore it is better to content oneself with finding constraints in some classes, and these classes must be chosen using additional considerations.

It was recently proposed a new method for finding differential constraints which uses linear determining equations. These equations are more general than the classical determining equations for Lie generators [6] and depend on some parameters. Given an evolution equation

$$u_t = F(t, x, u, u_1, \dots, u_n), \tag{1.3}$$

where $u_k = \frac{\partial^k u}{\partial x^k}$, then according to [7] the linear determining equation corresponding to (1.3) is of the form

$$D_t(h) = \sum_{i=0}^{n} \sum_{k=0}^{i} b_{ik} D_x^{i-k}(F_{u_{n-k}}) D_x^{n-i}(h), \qquad b_{ik} \in R. \tag{1.4}$$

Here and throughout D_t, D_x are the operators of total differentiation with respect to t and x. Equality (1.4) must hold for all solutions of (1.3). The function h may depends on $t, x, u, u_1, \dots, u_p$. The number p is called the order of the solution of the equation (1.4). If we have some solution h, the corresponding differential constraint is

$$h = 0. \tag{1.5}$$

It was also shown in [7] that the equations (1.4) and (1.5) constitute the compatible system.

The organization of this chapter as follows. In section 2 we introduce the requisite concepts and derive a nonlinear determining equation. Linearizing and

slightly this equation modifying one we obtain the linear determining equation. In section 3 we focus on the solutions of the second and third orders to the linear determining equation for Gibbons-Tsarev's equation [8]

$$u_{tt} = u_x u_{tx} - u_t u_{xx} + 1. \tag{1.6}$$

This gives the corresponding differential constraints and allow us to find some exact solutions of (1.6). Section 4 is devoted to systems of reaction-diffusion equations

$$u_t = (u^k v^l u_x)_x + f_1(u, v),$$
$$v_t = d_1(u^m v^n v_x)_x + f_2(u, v).$$

In the final section the invariant solutions under an involutive distribution are discussed. We consider the problem of finding involutive distributions that enable us to obtain invariant solutions to evolution equations.

2. Linear determining equations for differential constraints.

We consider a system S of evolution equations:

$$u^i_t = F^i(t, x, u^1, \ldots, u^m, u^1_x, u^2_x, \ldots), \quad i = 1, \ldots, m. \tag{2.1}$$

The right-hand sides of (2.1) can depend on the derivatives of arbitrary orders with respect to x. For simplicity we assume that the F^i are smooth functions. We supplement S by differential constraints H:

$$h_j(t, x, u^1, \ldots, u^m, u^1_x, u^2_x, \ldots) = 0, \quad j = 1, \ldots, p \tag{2.2}$$

where $p \leq m$. The functions h_i can depends on derivatives of arbitrary orders with respect to x, but contain no t-derivatives.

According to [11], the differential constraints H define an invariant manifold of the system S if

$$D_t(h_j)_{|[S]\cap[H]} = 0, \quad j = 1, \ldots, p. \tag{2.3}$$

We denote by $[S]$ the equations (2.1) and their differential consequences with respect to x. The constraint (2.2) and its differential consequences with respect to x are denoted by $[H]$.

If there are m differential constraints which are all resolved with respect to highest derivatives of all functions $u^1, ..., u^m$, then, as shown in [18] the invariance condition (2.3) yields the existence of a smooth solution of the system (2.1), (2.2).

For simplicity we now consider an equation of the second order

$$u_t = F(t, x, u, u_1, u_2) \tag{2.4}$$

and the differential constrain

$$h(t, x, u, u_1, \ldots, u_n) = 0,$$

where $u_k = \frac{\partial^k u}{\partial x^k}$. We denote by $[E]$ the equation (2.4) and its differential consequences.

The invariance condition (2.3) is equivalent to the following nonlinear equation

$$\begin{aligned} D_t(h) \mid_{[E]} = F_{u_2} D_x^2(h) + \Big(F_{u_1} + n D_x(F_{u_2})\Big) D_x(h) + \\ + \Big(F_u + n D_x(F_{u_1}) - h_{u_{n-1}} D_x(F_{u_2}) + \frac{n(n-1)}{2} D_x^2(F_{u_2}) + \\ + F_{u_2} h h_{u_{n-1} u_{n-1}} - 2F_{u_2} D_x(h_{u_{n-1}})\Big) h, \end{aligned} \tag{2.5}$$

with $n \geq 4$.

Indeed, it is easy to see that

$$D_t(h)_{|[E]} \simeq D_x^n(F) + h_{u_{n-1}} D_x^{n-1}(F) + h_{u_{n-2}} D_x^{n-2}(F). \tag{2.6}$$

Here and throughout we write $\alpha \simeq \beta$ to indicate that there are no terms including u_n, u_{n+1}, u_{n+2} in the difference $\alpha - \beta$. Since $n \geq 4$, the terms on the right-hand side of (2.6) can be represented as follows:

$$D_x^n(F) \simeq F_{u_2} u_{n+2} + [F_{u_1} + n D_x(F_{u_2})] u_{n+1} + \Big[F_u + n D_x(F_{u_1}) + \frac{n(n-1)}{2} D_x^2(F_{u_2})\Big] u_n,$$

$$h_{u_{n-1}} D_x^{n-1}(F) \simeq h_{u_{n-1}} \Big[u_n \Big(F_{u_1} + (n-1) D_x(F_{u_2})\Big) + u_{n+1} F_{u_2}\Big],$$

$$h_{u_{n-2}} D_x^{n-2}(F) \simeq u_n F_{u_2} h_{u_{n-2}}.$$

Hence (2.6) can be written as the relation

$$D_t(h)_{|[E]} \simeq F_{u_2}u_{n+2} + u_{n+1}[F_{u_1} + nD_x(F_{u_2}) + F_{u_2}h_{u_{n-1}}] + \Big[F_u +$$

$$+\frac{n(n-1)}{2}D_x^2(F_{u_2}) + nD_x(F_{u_1}) + h_{u_{n-1}}(F_{u_1} + nD_x(F_{u_2})) + F_{u_2}h_{u_{n-2}}\Big]u_n.$$

It is easy to see that

$$D_x(h) \simeq u_{n+1} + u_n h_{u_{n-1}},$$

$$D_x^2(h) \simeq u_{n+2} + u_{n+1}h_{u_{n-1}} + u_n[h_{u_{n-2}} + 2D_x(h_{u_{n-1}}) - u_n h_{u_{n-1}u_{n-1}}].$$

Hence the difference

$$D_t(h)_{|[E]} - F_{u_2}D_x^2(h) - [F_{u_1} + nD_x(F_{u_2})]D_x(h)$$

contains no terms with u_{n+2} and u_{n+1}. A direct calculation shows that there are no terms containing u_n in the expression for the function

$$\gamma = D_t(h)_{|[E]} - M(h),$$

where

$$M(h) = F_{u_2}D_x^2(h) + [F_{u_1} + nD_x(F_{u_2})]D_x(h) + \Big[F_u + nD_x(F_{u_1}) -$$

$$-h_{u_{n-1}}D_x(F_{u_2}) + \frac{n(n-1)}{2}D_x^2(F_{u_2}) + F_{u_2}hh_{u_{n-1}u_{n-1}} - 2F_{u_2}D_x(h_{u_{n-1}})\Big]h,$$

that is, $\gamma \simeq 0$. We claim that γ is equal to zero. Since H is an invariant manifold, it follows that

$$M(h) + \gamma = 0$$

on the set $[E] \cap [H]$. Clearly, $M(h)$ vanish there, therefore

$$\gamma_{|[E]\cap[H]} = 0.$$

Since γ is independent of u_t, $u_{tx}, \ldots$, we can rewrite the last equality as follows:

$$\gamma_{|[H]} = 0.$$

As shown above, $\gamma \simeq 0$, that is γ can depend only on $u_{n-1}, u_{n-2}, \ldots$. On the other hand, h depends on u_n. Hence γ is equal to zero.

It is clear that the equation (2.5) is difficult to solve, therefore, in place of the non-linear equation, we propose to use in the search for invariant manifolds linear equations of a similar kind. This leads to the following definition.

Definition. The equation of the form

$$D_t(h)_{|[E]} = F_{u_2} D_x^2(h) + \Big(c_1 F_{u_1} + c_2 D_x(F_{u_2})\Big) D_x(h) + \Big(c_3 F_u + c_4 D_x(F_{u_1}) + c_5 D_x^2(F_{u_2})\Big) h \tag{2.7}$$

is called the linear determining equation corresponding to (2.4). Here $c_1, \ldots, c_5$ are some constants.

It is easy to generalize the previous definition for case of equation (1.3).

Definition. A linear determining equation corresponding to (1.3) is the equation (1.4).

The nonlinear diffusion equation

$$u_t = (u^k u_x)_x + f(u)$$

have been considered in [12]. It is shown that the solutions of linear determining equations of the second and third orders exist only if f belongs to the special forms.

In particular the equation

$$u_t = (u^{-1/2} u_x)_x + mu - 2k\sqrt{u}, \qquad m, k \in R$$

is compatible with differential constrain

$$u_3 - \frac{5u_1u_2}{2u} + \frac{5u_1^3}{4u^2} + re^{-3mt/2}u^{5/2} + se^{mt/2}\sqrt{u} = 0, \qquad r, s \in R.$$

This leads to the following representation

$$u = -\frac{2X'T'}{(X+T)^2} e^{3mt/2}.$$

The functions $T(t)$ and $X(x)$ satisfy the ordinary differential equations

$$(X')^3 = (c_3X^3 + c_2X^2 + c_1X + c_0)^2,$$

$$(T')^3 = A(-c_3T^3 + c_2T^2 - c_1X + c_0)^2,$$

where c_3, c_2, c_1 and c_0 are arbitrary constants, $A = (-2r)^{1/3}$. It is possible to show that the functions X and T can be expressed by means of elliptic functions [12]. The others examples can be found in [12].

The linear determining equation corresponding to the equation

$$u_{tt} = F(t, x, u, u_1, \ldots, u_n) \tag{2.8}$$

is as follows:

$$D_{tt}(h)|_{[E]} = \sum_{i=0}^{n} \sum_{k=0}^{i} b_{ik} D_x^{i-k}(F_{u_{n-k}}) D_x^{n-i}(h), \qquad b_{ik} \in R, \tag{2.9}$$

where $[E]$ means the equation (2.8) and its differential consequences.

Example. Let us consider the equation

$$u_{tt} = u_{xx} + \frac{a}{t} u_t + \frac{b}{x} u_x + \sin(u), \tag{2.10}$$

It is easy to check that the symmetry group [6] for (2.1) is trivial. On the other hand, the linear determining equation

$$D_{tt}h = D_{xx}h + \frac{a}{t} D_t h + \frac{b}{x} D_x h + (\cos(u) - \frac{a}{t^2} - \frac{b}{x^2})h = 0$$

corresponding to (2.10) has the solution $h = x u_t + t u_x$. Differential constraint

$$h = x u_t + t u_x = 0$$

gives the following representation

$$u = V(x^2 - t^2) \tag{2.11}$$

for a solution of (2.10). Substituting (2.11) into (2.10) we have the ordinary differential equation

$$4zV'' + 2(b - a + 2)V' + \sin(V) = 0.$$

3. Gibbons-Tsarev's equation.

In this section we will consider the Gibbons-Tsarev equation [8]

$$z_{xx} + z_y z_{xy} - z_x z_{yy} + 1 = 0. \tag{3.1}$$

which arises in reductions of the Benney equation.

By (2.9), the linear determining equation has the form

$$D_x^2 h + z_y D_x D_y h - z_x D_y^2 h + b_1 z_{yy} D_x h + b_2 z_{xy} D_y h = 0, \tag{3.2}$$

The constants b_1 and b_2 are to be determined together with the function h.

It can be shown that the equation (3.2) has a solution of the form

$$h = z_{yy} + g(x, y, z, z_x, z_y, z_{xx})$$

if and only if the function g is independent of z_{xx}. Therefore we shall start with solutions of the second order

$$h = z_{yy} + g(x, y, z, z_x, z_y). \tag{3.3}$$

Substituting (3.3) into (3.2) leads to an equation which includes derivatives of the third order. We can express all mixed derivatives by means of (3.1). Setting the coefficients of z_{xxx} and z_{yyy} equal to zero we obtain :

$$b_1 = 1, \qquad b_2 = -1.$$

The left-hand side of (3.2) is a polynomial with respect to z_{xx} and z_{yy}. This polynomial must identically vanish. Collecting similar terms we have the following equations:

$$p g_{pp} + q g_{pq} - g_{qq} + 2 g_p = 0, \tag{3.4}$$

$$\begin{gathered}(-q^2 - 2p) g_{pp} + 2 g_{qq} + q^2 g_{xp} + q(q^2 + 2p) g_{yp} + q^2(q^2 + 3p) g_{zp} - \\ -2q g_{xq} - q^2 g_{yq} - q(q^2 + 2p) g_{zq} + q g_y + q^2 g_z - 4 g_p = 0,\end{gathered} \tag{3.5}$$

$$\begin{gathered}2p^2 g_{pp} - (q^2 + 2p) g_{qq} + p q^2 g_{xp} - 2p^2 q g_{yp} - p^2 q^2 g_{zp} + \\ +q(q^2 + 2p) g_{xq} - p q^2 g_{yq} + 2p^2 q g_{zq} + q^2 g_x - p q g_y + 4p g_p = 0,\end{gathered} \tag{3.6}$$

$$\begin{gathered}p g_{pp} - g_{pq} - g_{qq} + q^2 g_{xp} - 2pq g_{yp} - p q^2 g_{zp} + 2q g_{xq} + q^2 g_{yq} + q(q^2 + 2p) g_{zq} - \\ -q^2(q^2 + 2p) g_{xz} + p q^3 g_{yz} - p^2 q^2 g_{zz} + 2 g_p - q^2 g_{xx} - q^3 g_{xy} + p q^2 g_{yy} - q g_y = 0,\end{gathered} \tag{3.7}$$

where $p = z_x$ and $q = z_y$.

It is possible to show that the general solution of the equations (3.4) – (3.7) is

$$h = z_{yy} + c_1(z_y^4 + (3z_x + 4x)z_y^2 + 3yz_y + (z_x + 2x)^2 + 2z) +$$
$$+ c_2(z_y^3 + (2z_x + 3x)z_y + 2y) + c_3(z_y^2 + z_x + 2x) + c_4 z_y + c_5.$$

Hence the differential constraint $h = 0$ is compatible with the Gibbons-Tsarev equation (3.1). In the case $c_1 = c_2 = c_3 = 0$ we obtain the differential constraint

$$z_{yy} + c_4 z_y + c_5 = 0. \tag{3.8}$$

From (3.8) we find the following representation

$$z = a_1 \exp(-c_4 y) - c_5 y / c_4 + a_2, \tag{3.9}$$

where a_1 and a_2 depend on x. Substituting (3.9) into (3.1) we derive two ordinary differential equations

$$a_2'' + 1 = 0, \qquad a_1'' + c_5 a_1' - c_4^2 a_1 a_2' = 0.$$

The first equation has the solution

$$a_2 = -x^2/2 + c_6 x + c_7. \qquad c_6, c_7 \in R$$

In this case the second equation is

$$a_1'' + c_5 a_1' + c_4^2 (x - c_6) a_1 = 0.$$

Setting $a_1 = \exp(-c_5 x/2) v(x)$ we obtain the equation

$$v'' + (A + Bx)v = 0, \qquad A, B \in R.$$

According to [9] the solutions of last equation can be expressed in terms of the Airy functions.

It can be shown that the linear determining equation (3.2) has the following solution of the third order

$$h = z_{yyy} + c_1(3z_y^5 + (10z_x + 12x)z_y^3 + 6yz_y^2 + (6z_x^2 + 18xz_x + 2z + 12x^2)z_y +$$
$$+ 4yz_x + 6xy) + c_2(5z_y^4 + (12z_x + 15x)z_y^2 + 6yz_y + 3z_x^2 + 10xz_x + 15/2x^2 +$$
$$+ z) + c_3(2z_y^3 + (3z_x + 4x)z_y + y) + c_4(3z_y^2 + 2z_x + 3x) + c_5 z_y + c_6.$$

The corresponding constants b_1 and b_2 in (3.2) are given by

$$b_1 = 2, \qquad b_2 = -2.$$

In the case $c_1 = c_2 = c_3 = c_6 = 0$ and $c_5 = -1$ the function h gives the differential constraint

$$z_{yyy} - z_y = 0. \tag{3.10}$$

From (3.10) we obtain the following representation

$$z = s_1(x) + s_2(x)e^y + s_3(x)e^{-y}.$$

The functions $s_1(x)$, $s_2(x)$ and s_3 must satisfy the equations

$$s_2^{''} - s_1^{'} s_2 = 0,$$
$$s_1^{''} - 2s_3 s_2^{'} - 2s_2 s_3^{'} + 1 = 0,$$
$$s_3^{''} - s_1^{'} s_3 = 0.$$

If $s_3 = as_2$ then the last system reduces to the two equations

$$s_2^{''} - s_1^{'} s_2 = 0, \qquad s_1^{''} - 4as_2 s_2^{'} + 1 = 0, \quad a \in R. \tag{3.11}$$

Integrating the second equation, we find that

$$s_1^{'} = -t - b + 2as_2^2, \qquad b \in R.$$

We can insert this expression in (3.11) and obtain the second-order equation

$$s_2^{''} + (t + b - 2as_2^2)s_2 = 0$$

Using the transformations $t_1 = t + b$ and $w = \sqrt{a}s_2$, we take the equation in s_2 to the second Painleve equation [10]

$$w^{''} = 2w^3 + t_1 w.$$

The differential constraint

$$z_{yyy} = 0$$

leads to a solution of (3.1) which is expressed in terms of elementary functions.

4. The reaction-diffusion equations.

In this section we consider systems of the second order

$$u_t = (u^k v^l u_x)_x + f_1, \tag{4.1}$$

$$v_t = d_1(u^m v^n v_x)_x + f_2, \tag{4.2}$$

where k, l, m, n, and d_1 are arbitrary constants, the functions f_1 and f_2 depend on u and v. This model plays important role in various applications.

Using the method described in the section 2 it is possible to obtain the linear determining equations to the system (4.1) – (4.2). We give the equations in the final shape:

$$\begin{gathered}D_t(h) = u^k v^l D_x^2(h) + (b_1 k u^{k-1} v^l u_x + b_2 l u^k v^{l-1} v_x) D_x(h) + \\ + b_3 l u^k v^{l-1} u_x D_x(\beta) + \Big(b_4 f_{1u} + (b_4 + 2b_5 + b_6) k (u^{k-1} v^l u_{xx} + \\ + (k-1) u^{k-2} v^l u_x^2) + (b_4 + 3b_5 + b_6) k l u^{k-1} v^{l-1} u_x v_x + (b_5 + \\ + b_6) l (u^k v^{l-1} v_{xx} + (l-1) u^k v^{l-2} v_x^2)\Big) h + \Big(b_7 f_{1v} + \\ + b_8 l (u^k v^{l-1} u_{xx} + k u^{k-1} v^{l-1} u_x^2 + (l-1) u^k v^{l-2} u_x v_x)\Big)\beta,\end{gathered} \tag{4.3}$$

$$\begin{gathered}D_t(\beta) = d_1 u^m v^n D_x^2(\beta) + d_1(b_9 m u^{m-1} v^n u_x + b_{10} n u^m v^{n-1} v_x) D_x(\beta) + \\ + b_{11} d_1 m u^{m-1} v^n v_x D_x(h) + \Big(b_{12} f_{2u} + b_{13} d_1 m (u^{m-1} v^n v_{xx} + \\ + n u^{m-1} v^{n-1} v_x^2 + (m-1) u^{m-2} v^n u_x v_x)\Big) h + \Big(b_{14} f_{2v} + d_1 n (b_{14} + \\ + 2b_{15} + b_{16})(u^m v^{n-1} v_{xx} + (n-1) u^m v^{n-2} v_x^2) + d_1 (b_{14} + 3b_{15} + \\ + b_{16}) m n u^{m-1} v^{n-1} u_x v_x + d_1 m (b_{15} + b_{16})(u^{m-1} v^n v u_{xx} + \\ + (m-1) u^{m-2} v^n u_x^2)\Big)\beta.\end{gathered} \tag{4.4}$$

Here h and β are required functions, b_{ij} are some constants.

At first we seek for solutions of the second order to (4.3)-(4.4) of the form

$$h = u_{xx} + h_1(t, x, u, v, u_x, v_x), \;\; \beta = v_{xx} + \beta_1(t, x, u, v, u_x, v_x).$$

We do not itemize the systems (4.1)-(4.2) which lead to the simplest solutions

$$h = u_{xx}, \;\; \beta = v_{xx}$$

of the equations (4.3)-(4.4). Moreover, we did not include solutions generated by the symmetry groups.

The full list of nonlinear systems and corresponding solutions of linear determining equations is as follows:

$$u_t = (uu_x)_x + 2f_{11}u + f_{12}v + f_{13}, \qquad h_1 = u_{xx} + f_{11},$$
$$v_t = d_1 v_{xx} + f_{21}u - \frac{f_{21}f_{12}}{f_{11}}v + f_{23}, \qquad \beta_1 = v_{xx} + \frac{f_{11}^2}{f_{12}};$$

$$u_t = (uu_x)_x + 2f_{11}u + f_{12}v + f_{13}, \qquad h_2 = u_{xx} + f_{11},$$
$$v_t = d_1(uv_x)_x + f_{21}u + (3d_1f_{11} - 2f_{12})v + f_{23}, \qquad \beta_2 = v_{xx} + \frac{f_{11}^2}{f_{12}};$$

$$u_t = (uu_x)_x + \frac{f_{11}}{2}u^2 + f_{12}v - \frac{f_{21}f_{12}}{f_{11}}, \qquad h_3 = u_{xx} + \frac{f_{11}}{3}u,$$
$$v_t = -2(uv_x)_x + f_{21}u - f_{11}uv - \frac{f_{11}^2}{2f_{12}}u^3, \qquad \beta_3 = v_{xx} + \frac{1}{3f_{12}}\left(f_{11}f_{12}v + \frac{f_{11}^2}{2}u^2 - f_{21}f_{12}\right);$$

$$u_t = (uu_x)_x + f_{11}uv + f_{12}u^2 + f_{13}u + f_{14}, \qquad h_4 = u_{xx} + \frac{f_{11}}{3}v + \frac{2f_{12}}{3}u + \frac{f_{13}}{3},$$
$$v_t = -2(uv_x)_x - \frac{2f_{12}}{f_{11}}(f_{11}uv + f_{12}u^2 + f_{13}u + f_{14}), \qquad \beta_4 = 0;$$

$$u_t = (uu_x)_x + f_{11}u + f_1(v), \qquad h_5 = u_{xx} + f_{11},$$
$$v_t = \left(f_{21}u + f_{22} - \frac{f_{21}}{2f_{11}}f_1(v)\right)/f_1'(v), \qquad \beta_5 = v_{xx} + \frac{f_1''(v)}{f_1'(v)}v_x^2 + \frac{2f_{11}^2}{f_1'(v)};$$

$$u_t = (uu_x)_x + f_{11}u + f_{12}v + f_{13}, \qquad h_6 = u_{xx} + f_{11},$$
$$v_t = d_1(vv_x)_x + \frac{2f_{11}}{f_{12}^2}(6d_1 f_{11}^2 - f_{12}f_{22})u + \qquad \beta_6 = v_{xx} + \frac{2f_{11}^2}{f_{12}};$$
$$+f_{22}v + f_{23},$$

$$u_t = u_{xx} + f_{11}u - \frac{f_{22}f_{11}}{f_{21}}v + f_{13}, \qquad h_7 = u_{xx} + \frac{2f_{22}}{3},$$
$$v_t = d_1(uv_x)_x + d_1 f_{21}u + d_1 f_{22}v + f_{23}, \quad \beta_7 = v_{xx} + \frac{2f_{21}}{3};$$

$$u_t = u_{xx} + f_{11}u + f_{12}v + f_{13}, \qquad h_8 = u_{xx} + \frac{f_{12}}{3f_{11}^2 d_1}(f_{21}f_{12} - f_{22}f_{11}),$$
$$v_t = d_1(vv_x)_x + f_{21}u + f_{22}v + f_{23}, \qquad \beta_8 = v_{xx} - \frac{f_{21}f_{12} - f_{22}f_{11}}{3f_{11}d_1};$$

$$u_t = (vu_x)_x + f_{11}u + f_{12}v + f_{13}, \qquad h_9 = u_{xx} + \frac{f_{12}f_{21} - f_{11}f_{22}}{3f_{21}},$$
$$v_t = d_1 v_{xx} + f_{21}u + f_{22}v + f_{23}, \qquad \beta_9 = v_{xx} - \frac{f_{12}f_{21} - f_{11}f_{22}}{3f_{22}};$$

$$u_t = (vu_x)_x + f_{11}u + f_{12}v + f_{13}, \qquad h_{10} = u_{xx} + \frac{f_{11}f_{22} - f_{12}f_{21}}{3(d_1 f_{11} - f_{21})},$$
$$v_t = d_1(uv_x)_x + f_{21}u + f_{22}v + f_{23}, \qquad \beta_{10} = v_{xx} + \frac{f_{12}f_{21} - f_{11}f_{22}}{3(d_1 f_{12} - f_{22})};$$

$$u_t = (vu_x)_x + f_{11}u + f_{12}v + f_{13}, \qquad h_{11} = u_{xx} - \frac{f_{12}f_{22}}{d_1 f_{11} - 3f_{22}},$$
$$v_t = d_1(vv_x)_x - \frac{2f_{22}(d_1 f_{11} - 3f_{22})}{d_1 f_{12}}u, \quad \beta_{11} = v_{xx} + \frac{f_{22}}{d_1} + f_{22}v + f_{23};$$

$$u_t = (vu_x)_x + f_{11}u + \frac{5f_{11}f_{22}}{f_{21}}v + f_{13}, \quad h_{12} = u_{xx} + \frac{f_{11}f_{22}}{2f_{21}},$$
$$v_t = f_{21}u + f_{22}v + f_{23}, \qquad \beta_{12} = v_{xx} - \frac{f_{11}}{2}.$$

$$u_t = (v^l u_x)_x + f_{11}u + \frac{5f_{11}f_{23}}{2f_{21}}v^l + f_{13}, \qquad h_{13} = u_{xx} + \frac{f_{11}f_{23}}{2f_{21}},$$
$$v_t = f_{21}v^{1-l}u + f_{22}v^{1-l} + f_{23}v, \qquad \beta_{13} = v_{xx} + (l-1)\frac{v_x^2}{v} - \frac{f_{11}}{2l}v^{1-l},$$

$$u_t = (uu_x)_x + f_{11}u + f_{12}v^{1/4} + f_{13}, \qquad h_{14} = u_{xx},$$
$$v_t = d_1(v^{1/2}v_x)_x + f_{21}uv^{3/4} + f_{22}v^{3/4} + f_{23}v, \qquad \beta_{14} = v_{xx} - \frac{3v_x^2}{4v};$$

$$u_t = (uu_x)_x + f_{11}u + f_{12}v^{3/2} + f_{13}, \qquad h_{15} = u_{xx},$$
$$v_t = d_1(v^{1/2}v_x)_x + f_{21}uv^{-1/2} + f_{22}v^{-1/2} + f_{23}v, \quad \beta_{15} = v_{xx} + \frac{v_x^2}{2v};$$

$$u_t = (u^{1/2}u_x)_x + f_{11}u^{3/4}v + f_{12}u^{3/4} + f_{13}u, \qquad h_{16} = u_{xx} - \frac{3u_x^2}{4u},$$
$$v_t = d_1(vv_x)_x + f_{21}v + f_{22}u^{1/4} + f_{23}, \qquad \beta_{16} = v_{xx};$$

$$u_t = (u^{1/2}u_x)_x + f_{11}u^{-1/2}v + f_{12}u^{-1/2} + f_{13}u, \qquad h_{17} = u_{xx} + \frac{u_x^2}{2u},$$
$$v_t = d_1(vv_x)_x + f_{21}v + f_{22}u^{3/2} + f_{23}, \qquad \beta_{17} = v_{xx};$$

$$u_t = u_{xx} + f_{11}u + f_{12}v^{1/4} + f_{13}, \qquad h_{18} = u_{xx},$$
$$v_t = d_1(v^{1/2}v_x)_x + f_{21}uv^{3/4} + f_{22}v^{3/4} + f_{23}v, \qquad \beta_{18} = v_{xx} - \frac{3v_x^2}{4v};$$

$$u_t = u_{xx} + f_{11}u + f_{12}v^{3/2} + f_{13}, \qquad h_{19} = u_{xx},$$
$$v_t = d_1(v^{1/2}v_x)_x + f_{21}uv^{-1/2} + f_{22}v^{-1/2} + f_{23}v, \quad \beta_{19} = v_{xx} + \frac{v_x^2}{2v};$$

$$u_t = (u^{1/2}u_x)_x + f_{11}u^{3/4}v + f_{12}u^{3/4} + f_{13}u, \qquad h_{20} = u_{xx} - \frac{3u_x^2}{4u},$$
$$v_t = d_1v_{xx} + f_{21}v + f_{22}u^{1/4} + f_{23}, \qquad \beta_{20} = v_{xx};$$

$$u_t = (u^{1/2}u_x)_x + f_{11}u^{-1/2}v + f_{12}u^{-1/2} + f_{13}u, \qquad h_{21} = u_{xx} + \frac{u_x^2}{2u},$$
$$v_t = d_1v_{xx} + f_{21}v + f_{22}u^{3/2} + f_{23}, \qquad \beta_{21} = v_{xx};$$

$$u_t = (u^{1/2}u_x)_x + f_{11}u^{3/4}v^{1/4} + f_{12}u^{3/4} + f_{13}u, \qquad h_{22} = u_{xx} - \frac{3u_x^2}{4u},$$
$$v_t = d_1(v^{1/2}v_x)_x + f_{21}u^{1/4}v^{3/4} + f_{22}v^{3/4} + f_{23}v, \qquad \beta_{22} = v_{xx} - \frac{3v_x^2}{4v};$$

$$u_t = (u^{1/2}u_x)_x + f_{11}u^{3/4}v^{3/2} + f_{12}u^{3/4} + f_{13}u, \qquad h_{23} = u_{xx} - \frac{3u_x^2}{4u},$$
$$v_t = d_1(v^{1/2}v_x)_x + f_{21}u^{1/4}v^{-1/2} + f_{22}v^{-1/2} + f_{23}v, \quad \beta_{23} = v_{xx} + \frac{v_x^2}{2v};$$

$$u_t = (u^{1/2}u_x)_x + f_{11}u^{-1/2}v^{1/4} + f_{12}u^{-1/2} + f_{13}u, \qquad h_{24} = u_{xx} + \frac{u_x^2}{2u},$$
$$v_t = d_1(v^{1/2}v_x)_x + f_{21}u^{3/2}v^{3/4} + f_{22}v^{3/4} + f_{23}v, \qquad \beta_{24} = v_{xx} - \frac{3v_x^2}{4v};$$

$$u_t = (u^{1/2}u_x)_x + f_{11}u^{-1/2}v^{1/4} + f_{12}u^{-1/2} + f_{13}u, \qquad h_{25} = u_{xx} + \frac{u_x^2}{2u},$$
$$v_t = d_1(v^{1/2}v_x)_x + f_{21}u^{1/4}v^{-1/2} + f_{22}v^{-1/2} + f_{23}v, \quad \beta_{25} = v_{xx} + \frac{v_x^2}{2v};$$

$$u_t = (u^{1/2}u_x)_x + f_{11}u^{3/4}v + f_{12}u^{3/4} + f_{13}u, \qquad h_{26} = u_{xx} - \frac{3u_x^2}{4u},$$
$$v_t = d_1(u^{1/2}v_x)_x + f_{21}v + f_{22}u^{1/4} + f_{23}, \qquad \beta_{26} = v_{xx};$$

$$u_t = (v^{1/2}u_x)_x + f_{11}u + f_{12}v^{1/4} + f_{13}, \qquad h_{27} = u_{xx},$$
$$v_t = d_1(v^{1/2}v_x)_x + f_{21}uv^{3/4} + f_{22}v^{3/4} + f_{23}v, \qquad \beta_{27} = v_{xx} - \frac{3v_x^2}{4v};$$

$$u_t = (u^{1/2}u_x)_x + f_{11}u^{3/4}v + f_{12}u^{3/4} + f_{13}u, \qquad h_{28} = u_{xx} - \frac{3u_x^2}{4u},$$
$$v_t = d_1(u^{1/4}v_x)_x + f_{21}v + f_{22}u^{1/4} + f_{23}, \qquad \beta_{28} = 0;$$

$$u_t = (v^{1/4}u_x)_x + f_{11}u + f_{12}v^{1/4} + f_{13}, \qquad h_{29} = u_{xx},$$
$$v_t = d_1(v^{1/2}v_x)_x + f_{21}uv^{3/4} + f_{22}v^{3/4} + f_{23}v, \qquad \beta_{29} = v_{xx} - \frac{3v_x^2}{4v};$$

$$u_t = (u^{1/2}u_x)_x + f_{11}u^{-1/2}v + f_{12}u^{-1/2} + f_{13}u, \qquad h_{30} = u_{xx} + \frac{u_x^2}{2u},$$
$$v_t = d_1(u^{3/2}v_x)_x + f_{21}v + f_{22}u^{3/2} + f_{23}, \qquad \beta_{30} = v_{xx};$$

$$u_t = (v^{3/2}u_x)_x + f_{11}u + f_{12}v^{3/2} + f_{13}, \qquad h_{31} = u_{xx},$$
$$v_t = d_1(v^{1/2}v_x)_x + f_{21}uv^{-1/2} + f_{22}v^{-1/2} + f_{23}v, \quad \beta_{31} = v_{xx} + \frac{v_x^2}{2v};$$

$$u_t = (u^{1/2}u_x)_x + f_{11}u^{3/4}v^{1+a_1} + f_{12}u^{3/4} + f_{13}u, \quad h_{32} = u_{xx} - \frac{3u_x^2}{4u},$$
$$v_t = f_{21}u^{1/4}v^{-a_1} + f_{22}v^{-a_1} + f_{23}v, \qquad \beta_{32} = v_{xx} + a_1\frac{v_x^2}{v};$$

$$u_t = (u^{1/2}u_x)_x + f_{11}u^{-1/2}v^{1+a_1} + f_{12}u^{-1/2} + f_{13}u, \quad h_{33} = u_{xx} + \frac{u_x^2}{2u},$$
$$v_t = f_{21}u^{3/2}v^{-a_1} + f_{22}v^{-a_1} + f_{23}v, \qquad \beta_{33} = v_{xx} + a_1\frac{v_x^2}{v};$$

$$u_t = (v^{1/2}u_x)_x + (3a_1v^{1/2}/4 + f_{11})u + \qquad h_{34} = u_{xx} + \frac{a_1}{4}(u + f_{22}/f_{21}),$$
$$+\frac{3a_1f_{22}}{4f_{21}}v^{1/2} + \frac{f_{11}f_{22}}{f_{21}}),$$
$$v_t = f_{21}uv^{3/4} + f_{22}v^{3/4} + f_{23}, \qquad \beta_{34} = v_{xx} - \frac{3v_x^2}{4v} + a_1v.$$

$$u_t = u_{xx} + f_1(u,v), \qquad h_{35} = u_{xx} + f_1(u,v) + a_1u + a_2,$$
$$v_t = \frac{a_1f_{1u}(u,v)u + a_2f_{1u}(u,v) - a_1f_1(u,v)}{f_{1v}(u,v)}, \quad \beta_{35} = 0;$$

$$u_t = u_{xx} + uf_1(v) + a_1u\ln(u), \qquad h_{36} = u_{xx} + uf_1(v) + a_1u\ln(u) +$$
$$+a_2e^{a_1t}u + a_3,$$
$$v_t = a_3\frac{f_1(v) + a_1(\ln(u) + 1)}{uf_1'(v)}, \qquad \beta_{36} = 0;$$

$$u_t = u_{xx} + f_1\left(\frac{a_1v + a_2}{f_{21}e^{f_{23}u}}\right), \qquad h_{37} = u_{xx} + a_1u_x + f_1\left(\frac{a_1v + a_2}{f_{21}e^{f_{23}u}}\right) +$$
$$\frac{a_3x + a_4}{f_{23}},$$
$$v_t = f_{21}v + f_{22}e^{f_{23}u} + f_{24}, \qquad \beta_{37} = a_1v_x + f_{21}v + f_{22}e^{f_{23}u} +$$
$$+f_{24} + (a_3v + a_4)v + \frac{f_{24}(a_3v + a_4)}{f_{21}};$$

$$u_t = u_{xx} + f_{21}(a_2-1)u\ln(u) + uf_1\Big(\frac{v}{u^{a_1}}\Big),$$
$$v_t = f_{21}v + f_{22}u^{a_1},$$
$$h_{38} = u_{xx} + a_3e^{f_{21}(a_2-1)}u_x + f_{21}(a_2-1)u\ln(u) + uf_1\Big(\frac{v}{u^{a_1}}\Big)+$$
$$+\frac{f_{21}a_3}{2}(a_2-1)\Big(e^{f_{21}(a_2-1)t}x + \frac{1}{a_4}\Big(a_5e^{f_{21}(a_2-1)t} - \frac{2a_2a_4}{a_1a_3(a_2-1)}\Big)\Big)u,$$
$$\beta_{38} = a_3e^{f_{21}(a_2-1)t}v_x + f_{21}v + f_{22}u^{a_1} + \Big(\frac{a_1a_3(a_2-1)f_{21}}{2a_4}(a_4x+a_5)e^{f_{21}(a_2-1)t} - f_{21}\Big)v;$$

$$u_t = u_{xx} + f_1(u) + \frac{a_1}{d_1}((n+1)v - d_1v^{n+1}),$$
$$v_t = d_1(v^nv_x)_x + \frac{d_1}{n+1}(v^{n+1}+a_2u+a_3)f_1'(u) - \frac{a_2}{n+1}\Big((n+1-d_1n)v + \frac{d_1}{a_1}f_1(u)\Big),$$
$$h_{39} = u_{xx} + f_1(u) + \frac{a_1}{d_1}((n+1)v - d_1v^{n+1}) + a_2u + a_1v^{n+1} + a_3,$$
$$\beta_{39} = 0;$$

$$u_t = u_{xx} - \frac{a_1}{d_1u}(d_1u^2v^{n+1} - (n+1)v) + f_1(u),$$
$$v_t = d_1(u^2v^nv_x)_x + \frac{d_1}{n+1}\Big(\Big(v^{n+1} + \frac{a_2}{a_1}\Big)u^2 + \frac{a_3}{a_1}u\Big)f_1'(u) - \frac{d_1}{n+1}\Big(v^{n+1}+$$
$$+\frac{a_2}{a_1}\Big)uf_1(u) - \frac{d_1a_3}{n+1}uv^{n+1} - 2a_1v^{n+2} - 2a_2v - a_3\frac{v}{u},$$
$$h_{40} = u_{xx} - \frac{a_1}{d_1u}(d_1u^2v^{n+1} - (n+1)v) + f_1(u) + (a_1v^{n+1}+a_2)u + a_3,$$
$$\beta_{40} = 0,$$

where f_{ij} and a_k are arbitrary constants.

Moreover, we found all systems (4.1)-(4.2) which have solutions of the third order

$$h = u_{xxx} + h_1(t, x, u, v, u_1, v_1, u_2, v_2),$$
$$\beta = g_1v_{xxx} + \beta_1(t, x, u, v, u_1, v_1, u_2, v_2)$$

to the linear determining equations (4.3)-(4.4). The full list includes ten systems. We give these systems together with solutions of the linear determining equations:

$$u_t = (uu_x)_x + f_{11}u^2 + f_{12}u + f_{13}, \qquad h_{41} = u_{xxx} + f_{11}u_x/2,$$
$$v_t = d_1v_{xx} + f_{21}u + f_{22}v + f_{23}, \qquad \beta_{41} = v_{xxx} + f_{11}v_x/2;$$

$$u_t = (vu_x)_x + (f_{11}v + f_{12})u + f_{13}v + f_{14}, \qquad h_{42} = u_{xxx} + f_{11}u_x/2,$$
$$v_t = d_1v_{xx} + f_{21}u + f_{22}v + f_{23}, \qquad \beta_{42} = v_{xxx} + f_{11}v_x/2;$$

$$u_t = u_{xx} + f_{11}u + f_{12}v + f_{13}, \qquad h_{43} = u_{xxx} + f_{21}u_x/2,$$
$$v_t = d_1(vv_x)_x + d_1f_{21}v^2 + f_{22}v + f_{23}u + f_{24} \qquad \beta_{43} = v_{xxx} + f_{21}v_x/2;$$

$$u_t = u_{xx} + f_{11}u + f_{12}v + f_{13}, \qquad h_{44} = u_{xxx} + f_{21}u_x/2,$$
$$v_t = d_1(uv_x)_x + (f_{21}u + f_{22})v + f_{23}u + f_{24}, \qquad \beta_{44} = v_{xxx} + f_{21}v_x/2;$$

$$u_t = (uu_x)_x + f_{21}u^2 + f_{12}u + f_{13}v + f_{14}, \qquad h_{45} = u_{xxx} + f_{21}u_x/2,$$
$$v_t = d_1(uv_x)_x + (d_1f_{21}u + f_{22})v + f_{23}u + f_{24}, \qquad \beta_{45} = v_{xxx} + f_{21}v_x/2;$$

$$u_t = (vu_x)_x + (f_{21}v + f_{12})u + f_{13}u + f_{14}, \qquad h_{46} = u_{xxx} + f_{21}u_x/2,$$
$$v_t = d_1(vv_x)_x + d_1f_{21}v^2 + f_{22}v + f_{23}u + f_{24}, \qquad \beta_{46} = v_{xxx} + f_{21}v_x/2;$$

$$u_t = (uu_x)_x + f_{21}u^2 + f_{12}u + f_{13}v + f_{14}, \qquad h_{47} = u_{xxx} + f_{21}u_x/2,$$
$$v_t = d_1(vv_x)_x + d_1f_{21}v^2 + f_{22}v + f_{23}u + f_{24}, \qquad \beta_{47} = v_{xxx} + f_{21}v_x/2;$$

$$u_t = (vu_x)_x + (f_{21}u + f_{12})v + f_{13}u + f_{14}, \qquad h_{48} = u_{xxx} + f_{21}u_x/2,$$
$$v_t = d_1(uv_x)_x + (d_1f_{21}u + f_{22})v + f_{23}u + f_{24}, \qquad \beta_{48} = v_{xxx} + f_{21}v_x/2;$$

$$u_t = (uu_x)_x + f_{11}u^2 + f_{12}u + f_1(v), \qquad h_{49} = u_{xxx} + f_{11}u_x/2,$$
$$v_t = (f_{21}u + f_{22}f_1(v) + f_{23})/f_1'(v), \qquad \beta_{49} = v_{xxx} + 3\frac{f_1''(v)}{f_1'(v)}v_{xx}v_x + \frac{f_1'''(v)}{f_1'(v)}v_x^3 + f_{11}v_x/2;$$

$$u_t = (vu_x)_x + (f_{11}v + f_{12})u + f_{13}v + f_{14}, \qquad h_{50} = u_{xxx} + f_{11}u_x/2,$$
$$v_t = f_{21}u + f_{22}v + f_{23}, \qquad \beta_{50} = v_{xxx} + f_{11}v_x/2.$$

As some illustration of the basic method of computing solutions, we consider the system

$$u_t = (uu_x)_x + 2u^2 - 3u + v, \tag{4.5}$$

$$v_t = (vv_x)_x + 2v^2 - 2v + u, \tag{4.6}$$

with differential constraints

$$u_{xxx} + u_x = 0, \quad v_{xxx} + v_x = 0., \tag{4.7}$$

It follows from (4.7) that the functions u and v have the representation

$$u = u_1(t)\sin(x) + u_2(t)\cos(x) + u_3(t), \;\; v = v_1(t)\sin(x) + v_2(t)\cos(x) + v_3(t).$$

Substituting this representation into the system (4.5)-(4.6), we obtain ordinary differential equations

$$\begin{aligned} u_1' &= 3u_1(u_3 - 1) + v_1, \\ u_2' &= 3u_2(u_3 - 1) + v_2, \\ u_3' &= u_3(2u_3 - 3) + u_1^2 + u_2^2 + v_3, \\ v_1' &= v_1(3v_3 - 2) + u_1, \\ v_2' &= v_2(3v_3 - 2) + u_2, \\ v_3' &= 2v_3(v_3 - 1) + v_1^2 + v_2^2 + u_3. \end{aligned}$$

5. Invariant solutions under involutive distributions.

In this section we introduce invariant solutions under involutive distributions. Suppose that a collection of p vector fields

$$X_s = \sum_{i=1}^{n} \xi_s^i(x)\partial_{x_i}$$

is given on an open set $U \subset R^n$. If this collection is linearly disconnected, i.e., the rank of the matrix $|\xi_s^i(x)|$ equals p for all $x \in U$ and satisfies the involution condition

$$[X_i, X_j] = \sum_{k=1}^{p} c_{ij}^k(x)X_k, \qquad \forall\ 1 \le i, j \le p, \tag{5.1}$$

where c_{ij}^k are smooth functions, then this collection generates an involutive p-dimensional distribution D_p. A collection of vector fields with these properties is called an involutive basis or just a basis. It is well known that a distribution D_p is involutive if and only if it possesses at least one involutive basis.

Definition. A solution $u = \varphi$ to a system of partial differential equation E is invariant under an involutive distribution D_p if D_p is tangent to the manifold $S = \{(x, u) : \ \ u = \varphi(x)\}$. Obviously, the invariance of a solution under D_p amounts to its invariance under the operators of an arbitrary involutive basis for D_p.

Now, consider the system of evolution equations

$$u_t^i = F^i(t, x, u, u_\alpha), \quad i = 1, ..., m, \tag{5.2}$$

where t and $x = (x_1, \ldots, x_n)$ are independent variables, $u^1, ..., u^m$ are functions, $u = (u^1, ..., u^m)$, and u_α stands for various partial derivatives with respect to $x_1, \ldots, x_n$. Denote the total derivatives with respect to t and x_i by the symbols D_t and D_{x_i}.

Let $J^k(U, R^m)$ be the space of k-jets on $U \subset R^n$. Recall that a manifold $H \subset J^k(R^{n+1}, R^m)$, defined by the equations

$$h^j(t, x, u, u_\beta) = 0, \quad j = 1, ..., s, \tag{5.3}$$

is an invariant manifold for (5.2) if the following identity holds on the set $[E] \bigcap [H]$:

$$D_t h^j = 0.$$

Here $[E]$ and $[H]$ stand for the differential consequences of (5.2) and (5.3) with respect to $x_1, ..., x_n$. Denote the involutive distribution generated by vector fields $X_1, ..., X_r$ by $< X_1, ..., X_r >$.

Lemma 1. Suppose that vector fields

$$X_k = \sum_{i=1}^{n} \xi_k^i(t, x, u)\partial_{x_i} + \sum_{j=1}^{m} \eta_k^j(t, x, u)\partial_{u^j}, \ k = 1, ..., n, \tag{5.4}$$

generate an involutive distribution and that $det(\xi_k^i) \neq 0$. If the manifold defined by the equations

$$h_k^j = \sum_{i=1}^{n} \xi_k^i u_{x_i}^j - \eta_k^j = 0, \quad 1 \le j \le m, 1 \le k \le n, \tag{5.5}$$

is invariant with respect to (5.2) then system (5.2) has invariant solutions under this involutive distribution.

Proof. Write down the collection of the fields $X_1, ..., X_n$ in vector form as follows:

$$X = \xi \partial_x + \eta \partial_u.$$

Acting by the matrix ξ^{-1} on X, we obtain the involutive collection

$$Z = \partial_x + \tilde{\eta} \partial_u,$$

where $\tilde{\eta} = \xi^{-1} \eta$. The distribution $< Z_1, ..., Z_n >$ is involutive.

The invariant solutions under $< X_1, ..., X_n >$ must satisfy (5.5). The invariant solutions under $< Z_1, ..., Z_n >$ must satisfy the equations

$$u^j_{x_k} = \tilde{\eta}^j_k(t, x, u). \tag{5.6}$$

Obviously, (5.5) and (5.6) have the same solutions. Since Z is an involution distribution, the Poisson bracket $[Z_i, Z_k]$ vanishes. Consequently, we have

$$Z_i(\tilde{\eta}^j_k) = Z_k(\tilde{\eta}^j_i),$$

which means that the consistency conditions for (5.6) are satisfied.

Using (5.6) and inserting the derivatives of the functions u^j with respect to x_k in the right-hand side of (5.2), we come to the system

$$u^j_t = G^j(t, x, u), \qquad j = 1, ..., m. \tag{5.7}$$

By the Frobenius theorem, the system of (5.6) and (5.7) is compatible if the relations

$$D_{x_k} G^j = D_t \tilde{\eta}^j_k, \qquad j = 1, ..., m; \; k = 1, ..., n \tag{5.8}$$

are valid by virtue of (5.7) and (5.8). Validity of these conditions follows from the invariance of (5.5) with respect to (5.2). Indeed, this invariance means that

$$D_t \left(u^j_{x_k} - \tilde{\eta}^j_k \right) = D_{x_k} F^j - D_t \tilde{\eta}^j_k = 0. \tag{5.9}$$

Inserting the derivatives with respect to x_k in (5.9), we see that (5.9) coincides with (5.8).

Remark. If an involutive distribution is generated by analytic vector fields $X_1, ..., X_p$, where $p < n$, (5.2) is a system of first-order equations with analytic

right-hand sides, and the rank of the matrix $\left(\xi_k^i\right)$ equals p, then (5.2) has an invariant solution relative to $X_1, ..., X_p$. The proof is carried out by the above scheme, but instead of the Frobenius theorem we should use the Riquier theorem on the existence of analytic solutions to an autonomous system with analytic right-hand sides [5].

To exemplify the application of a distribution to constructing solutions, consider the equation

$$u_t = \Delta \ln u, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{5.10}$$

which arises in various application [13, 14] and possesses an infinite-dimensional algebra of point symmetries [15]. Some exact solutions to this equation can be found in [16, 17]. We give a solution to this equation which is invariant relative to the pair of commuting operators

$$X_1 = \partial_x - (u^2 + (tu^2 - xu^2 + u)\tan(t))\partial_u,$$

$$X_2 = \partial_y - (tu^2 + u - xu^2)\partial_u.$$

The corresponding manifold for these vector fields is

$$u_x + u^2 + (tu^2 - xu^2 + u)\tan(t) = 0, \tag{5.11}$$

$$u_y + tu^2 + u - xu^2 = 0. \tag{5.12}$$

It is easy to verify that this is an invariant manifold for (5.10). Note that the vector fields X_1 and X_2 do not belong to the algebra of symmetries of (5.10). The general solution to (5.11),(5.12) and (5.10) has the form

$$u = \frac{1}{A[\exp((x-t)\tan(t) + y]\cos t + x - t}, \quad A \in R.$$

To use vector fields and distributions, we need a method for finding them. The classical approach to constructing vector fields relative to which the given differential equations are invariant was proposed by S. Lie. A modern exposition with many examples and new results was given by L. V. Ovsyannikov [6].

A determining equation like (2.5) enables us to find differential constraints compatible with the original equation. In the case of differential equations in more than two independent variables, we can propose systems of defining equations which would enable us to find involutive distributions.

Consider the system of involution equations (5.2) and the manifold in $J^1(U, R^m)$ defined by

$$h^i_j = u^i_{x_j} + g^i_j(t, x, u) = 0, \tag{5.13}$$

where $i = 1, ..., m$, and $j = 1, ..., n$.

Theorem. Suppose that the manifold (5.13) is invariant under the system (5.2) whose right-hand sides are polynomials in derivatives whose coefficients depend on $t, x_1, ..., x_n$ and $u^1, ..., u^m$. Then the functions h^i_j satisfy the following system:

$$D_t h^i_j + m_{ij}(h)|_{[E]} = 0,\ 1 \le i \le m,\ 1 \le j \le n. \tag{5.14}$$

Here $m_{ij}(h)$ is some operator representing a polynomial in h^k_l, $D_{x_1} h^k_l, ..., D_{x_n} h^k_l, ..., D^\alpha h^k_l$ $(k = 1, ..., m,\ l = 1, ..., n)$. The operators $m_{ij}(h)$ vanish whenever all h^k_l are zero.

Proof. We first show that the total derivative of h^i_j with respect to t is representable as

$$D_t h^i_j = m_{ij}(h) + \gamma_{ij}, \tag{5.15}$$

where m_{ij} are operators whose shape is described in the theorem and γ_{ij} are functions which may depend only on t, x and u.

The following identities are valid on $[E]$:

$$D_t h^i_j = D_{x_j} F^i + \frac{\partial g^i_j}{\partial t} + \sum_{k=1}^{m} F^k \frac{\partial g^i_j}{\partial u^k}. \tag{5.16}$$

Let $\dfrac{\partial^{|s|} u^k}{\partial x_1^{s_1} \cdots \partial x_n^{s_n}}$ be a derivative of maximal order on the right-hand side of (5.16) and $s_p \neq 0$ for some p. By (5.16) and the assumptions of the theorem, this derivative enters (5.16) polynomially. Using (5.13), we can write down this derivative as follows:

$$D^{s_1}_{x_1} \cdots D^{s_p - 1}_{x_p} \cdots D^{s_n}_{x_n}(h^k_p) - D^{s_1}_{x_1} \cdots D^{s_p - 1}_{x_p} \cdots D^{s_n}_{x_n}(g^k_p).$$

Note that the second summand involves no derivatives of order $|s|$ and is a polynomial in derivatives. Thus, all derivatives of maximal order on the right-hand side of (5.16) can be expressed in terms of the total derivatives of the functions $h^r_q (r = 1, ..., m$, and $q = 1, ..., n)$. Afterwards, it is possible to express the derivatives of order $|s| - 1$, etc. down to the first-order derivatives.

We are left with demonstrating that the functions γ_{ij} in (5.15) are all zero. By the conditions of the theorem, the manifold (5.13) is an invariant manifold for (5.2). Consequently, the following identity holds on $[E]\cap[H]$:

$$m_{ij}(h) + \gamma_{ij} = D_t h^i_j = 0.$$

Since the m_{ij}'s vanish on $[H]$, the functions γ_{ij} are zero on $[E]\cap[H]$. Once the m_{ij}'s are independent of the derivatives of the functions u^k, all m_{ij} are identically zero.

Remark. As we see from the proof of the theorem, the choice of the operators m_{ij} could be not uniquely defined.

For example, consider the second-order equation in three independent variables:

$$u_t = G \equiv F^1 u_{xx} + F^2 u_{yy} + F^3 u_x^2 + F^4 u_y^2 + F^5, \tag{5.17}$$

where F^i are some functions depending on u. Suppose that

$$h_1 \equiv u_x + g_1(t,x,y,u) = 0,\ \ h_2 \equiv u_y + g_2(t,x,y,u) = 0 \tag{5.18}$$

define an invariant manifold for (5.17). To derive a system of determining equations like (5.14), we express the derivatives $D_t h_1$ and $D_t h_2$ in terms of $h_i, D_x h_i,\ D_y h_i,\ D_x^2 h_i, D_x D_y h_i$, and $D_y^2 h_i$ $(i = 1,2)$. By (5.17), the following holds:

$$D_t h_1 = D_x G + \frac{\partial g_1}{\partial t} + \frac{\partial g_1}{\partial u} G.$$

It is easy to verify that the right-hand side of the last equality is representable as

$$m_{11}(h_1,h_2) = G_{u_{xx}} D_x^2 h_1 + G_{u_{yy}} D_y^2 h_1 + [G_{u_x} + D_x(G_{u_{xx}})] D_x h_1 + G_{u_y} D_y h_1 +$$

$$+ D_x(G_{u_{yy}}) D_y h_2 + [G_u - D_x^2(G_{u_{xx}}) - D_y^2(G_{u_{yy}}) + r_1] h_1 + s_1 h_2 + \gamma_1, \tag{5.19}$$

where r_1, s_1, and γ_1 are functions depending on h_1, h_2, and G. Since (5.18) is an invariant manifold, the function γ_1 equals 0. Consequently, the first defining equation has the form

$$D_t h_1 = m_{11}(h_1, h_2).$$

To obtain the second defining equation

$$D_t h_2 = m_{12}(h_1, h_2),$$

we should replace h_1 in (5.12) with h_2, x with y, r_1 with r_2, and s_1 with s_2. The following lemma asserts that, under some conditions, solutions to equations like (3.13) enable us to construct differential constraints compatible with the system of evolution equations (5.2). It is worth to note that the form of the operators m_{ij} is inessential, provided that only $m_{ij}(0) = 0$.

Lemma 2. Suppose that the functions

$$h^i_j = \sum_{s=1}^{n} \xi^s_j(t, x, u) u^i_{x_s} - g^i_j(t, x, u)$$

satisfy a system like (5.14) on $[E]$ with $m_{ij}(0) = 0$. If the vector fields

$$X_j = \sum_{s=1}^{n} \xi^s_j \partial_{x_s} + \sum_{i=1}^{m} g^i_j \partial_{u_i}, \ j = 1, ..., n$$

generate an involutive distribution and $\det(\xi^s_j) \neq 0$ then there is a solution to the system of (5.2) and the equations

$$h^i_j = 0, \qquad i = 1, ..., m, \ j = 1, ..., n. \tag{5.20}$$

Proof. Since the functions h^i_j satisfy (5.14), in view of $m_{ij}(0) = 0$ (5.20) defines an invariant manifold for (5.2). To complete the proof, it suffices to refer to Lemma 1.

Finding solutions to general nonlinear equations (5.14) might represent a very complicated problem. To simplify the problem, we remove all terms non-linear in h^k_l from the operators m_{ij} as it was done above in the case of an evolution equation with one space variable. In result, we obtain some linear equation

$$D_t h^i_j + l_{ij}(h) = 0.$$

As we done above, multiply the coefficients of the operators l_{ij} by undetermined constants and write down the resultant equations as

$$D_t h^i_j + L_{ij}(h) = 0 \tag{5.21}$$

calling them linear determining equations (LDEs). For example, the LDEs for (5.17) have the form

$$D_t h_1 = L_{11}(h_1, h_2) \equiv a_1 G_{u_{xx}} D^2_x h_1 + a_2 G_{u_{yy}} D^2_y h_1 +$$

$$+[a_3 G_{u_x} + a_4 D_x(G_{u_{xx}})]D_x h_1 + a_5 G_{u_y} D_y h_1 + a_6 D_x(G_{u_{yy}}) D_y h_2 +$$
$$+[a_7 G_u + a_8 D_x^2(G_{u_{xx}}) + a_9 D_y^2(G_{u_{yy}})]h_1, \tag{5.22}$$
$$D_t h_2 = L_{12}(h_1, h_2),$$

where $L_{12}(h_1, h_2)$ is obtained from $L_{11}(h_1, h_2)$ by replacing h_1 with h_2, x with y, and a_i with b_i.

Although the above arguments were for systems of evolution equations, we can try to extend them to a more general situation. Assume given a system

$$n_i(u) = F^i(t, x, u, u_\alpha), \qquad i = 1, ..., m,$$

where n_i are linear differential operators with constant coefficients and the right-hand sides are similar to those in the case of evolution systems (5.2). To find the functions h_j^i, we suggest using the following equation in place of (5.21):

$$N_i(h_j^i) + L_{ij}(h) = 0, \tag{5.23}$$

where the operators N_i are obtained from n_i by replacing partial derivatives with total derivatives. Alongside (5.23), it is useful to introduce the following analog of B-defining equations [18]:

$$N_i(h_j^i) + L_{ij}(h) + \sum_{\substack{1\le l\le m \\ 1\le k\le n}} b_{lj}^{ki} h_k^l = 0, \tag{5.24}$$

where $1 \le i \le m, 1 \le j \le n$, and b_{lj}^{ki} are functions that may depend on t, x, and u.

We call equations of the form (3.23) quasilinear determining equations (QDEs). We exhibit an example of QDEs in finding involutive distributions. Consider one of the nonlinear dispersion models describing the propagation of long two-dimensional waves [22]:

$$\eta_{tt} = gd\Delta\eta + \frac{d^2}{3}\Delta\eta_{tt} + \frac{3}{2}g\Delta\eta^2,$$

where $\eta(t, x, y)$ is the deviation of a fluid from an equilibrium state, d is the depth of an unperturbed fluid, and g is the free fall acceleration. By translations and dilations, we can reduce this equation to the form

$$u_{tt} - \Delta(u_{tt}) - u\triangle u - (\nabla u)^2 = 0. \tag{5.25}$$

In accordance with the above method, the QLEs for (5.25) have the form

$$D_t^2 h_1 - D_t^2 D_x^2 h_1 - D_t^2 D_y^2 h_1 + a_1 u(D_x^2 h_1 + D_y^2 h_1) + a_2 u_x D_x h_1 + a_3 u_y D_y h_1$$

$$+a_4 u_x D_y h_2 + (a_5 \Delta u + a_6 u_{xx} + a_7 u_{yy} + r_1) h_1 + q_1 h_2 = 0, \tag{5.26}$$

$$D_t^2 h_2 - D_t^2 D_x^2 h_2 - D_t^2 D_y^2 h_2 + b_1 u(D_x^2 h_2 + D_y^2 h_2) + b_2 u_y D_y h_2 + b_3 u_x D_x h_2$$

$$+b_4 u_y D_x h_1 + (b_5 \Delta u + b_6 u_{xx} + b_7 u_{yy} + r_2) h_2 + q_2 h_1 = 0, \tag{5.27}$$

where a_i and b_i are constants, and r_j and q_j are functions which may depend on t, x, y, and u and which should be found together with h_1 and h_2. The scheme for solving (5.26) and (5.27) is completely analogous to the standard scheme of the group analysis of differential equations [6, 19]. For this reason, we omit all intermediate computations and set forth only the final results.

If h_1 and h_2 are sought in the form corresponding to the point symmetries

$$h_1 = \xi_1^1 u_t + \xi_2^1 u_x + \xi_3^1 u_y + \eta^1,$$

$$h_2 = \xi_1^2 u_t + \xi_2^2 u_x + \xi_3^2 u_y + \eta^2,$$

where ξ^i and η^j are functions of t, x, y, and u, then under the condition $(\xi_1^1)^2 + (\xi_3^1)^2 + (\xi_1^2)^2 + (\xi_2^2)^2 \neq 0$ equations (5.26) and (5.27) can be shown to have solutions leading only to admissible operators for (5.25). There appear new solutions only when

$$h_1 = u_x + g_1(t, x, y, u), \;\; h_2 = u_y + g_2(t, x, y, u).$$

The final form of g_1 and g_2 is as follows:

$$g_1 = s_1 x + s_2 y + s_3, \;\; g_2 = s_2 x + s_4 y + s_5.$$

Moreover, the functions s_i $(i = 1, ..., 5)$ depend only on t and satisfy the following system of five second-order differential equations:

$$s_1'' + 3s_1^2 + s_1 s_4 + 2s_2^2 = 0,$$

$$s_2'' + 3s_1 s_2 + 3s_2 s_4 = 0,$$

$$s_3'' + 3s_1 s_3 + 2s_2 s_5 + s_3 s_4 = 0,$$

$$s_4'' + s_1 s_4 + 2s_2^2 + 3s_4^2 = 0,$$

$$s_5'' + s_1 s_5 + 2 s_2 s_3 + 3 s_4 s_5 = 0.$$

For completeness of exposition, we write down the constants a_i and b_i ($i = 1, ..., 7$) and the functions r_j and q_j ($j = 1, 2$) in (3.26) and (3.27) corresponding to g_1 and g_2:

$$a_1 = b_1 = a_4 = b_4 = -1, \ a_2 = b_2 = a_3 = b_3 = -3,$$

$$a_5 = a_6 = a_7 = b_5 = b_6 = b_7 = 0,$$

$$r_1 = 3s_1 + s_4, \ r_2 = s_1 + 3s_4, \ q_1 = 2s_1, \ q_2 = 2s_2.$$

The functions h_1 and h_2 generate the differential constraints

$$u_x + s_1 x + s_2 y + s_3 = 0,$$

$$u_y + s_2 x + s_4 y + s_5 = 0.$$

These constraints enable us to find the following representation for a solution to (5.25):

$$u = \frac{-s_1 x^2}{2} - s_2 xy - \frac{s_4 y^2}{2} - s_3 x - s_5 y + s_6.$$

Inserting this in (5.25), we obtain the following equation for s_6:

$$s_6'' = 3s_1^2 + 2s_1 s_4 - s_1 s_6 + 4s_2^2 + s_3^2 + 3s_4^2 - s_4 s_6 + s_5^2.$$

The system of the six differential equations in the six functions s_i deserves further study. For example, it would be interesting to find a solution expressible via elementary functions.

Acknowledgement

This work was supported in part by RFBR, project 01 - 01 - 00850.

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